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Calculating Fragmentation Functions from Definitions

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Abstract:

Fragmentation functions for hadrons composed of heavy quarks are calculated directly from the definitions given by Collins and Soper and are compared with those calculated in another way. A new fragmentation function for a P-wave meson is also obtained and the singularity arising at the leading order is discussed.

Recently it has been shown[1,2,3,4] that the fragmentation functions for a hadron as a bound-state of two heavy quarks can be calculated perturbatively. The reason such calculations are possible is that the bound-states can be described well by the nonrelativistic wavefunctions and hence the effect of long distance in the fragmentaion can be factorized into the wavefunction, expressed through the radial wavefunction at the origin. Although it is realized[2,4] that the fragmentaion functions are universal, the fragmentation functions calculated in these works are extracted from specific processes. It should be noted that in QCD the definitions of the fragmentaion functions (or decay functions) are already given by Collins and Soper[5] and these functions should be universal and independent of pocesses. The so called factorization thorem is based on such definitions(for the factorization theorem see [6] and references cited there). It is interesting to calculate fragmentation functions directly from these defintions and to compare those obtained previously. On the other hand, it is also convenient to work with the defintions for calculating the high order corrections. In this letter we will calculate fragmentation functions from the definitions.

To give the definitions for a fragmentaion function it is convenient to work in the light-cone coordinate system. In this coordinate system a 4-vector p is expressed as $p^\mu = (p^+, p^-, \mathbf{p_T})$, with $p^+ = (p^0 + p^3)/\sqrt{2}$, $p^- = (p^0 - p^3)/\sqrt{2}$. Introducing a vector n with $n^\mu = (0, 1, \mathbf{0_T})$, the fragmentation functions for a spinless hadron H are defined as[5]:

$$\begin{aligned}
D_{H/Q}(z) &= \frac{z}{4\pi} \int dx^- e^{-iP^+ x^- / z} \frac{1}{3} \text{Tr}_{\text{color}} \frac{1}{2} \text{Tr}_{\text{Dirac}} \{n \cdot \gamma < 0 | Q(0) \\
&\quad \bar{P} \exp\{-ig_s \int_0^\infty d\lambda n \cdot G^T(\lambda n^\mu)\} a_H^\dagger(P^+, \mathbf{0_T}) a_H(P^+, \mathbf{0_T}) \\
&\quad P \exp\{ig_s \int_{x^-}^\infty d\lambda n \cdot G^T(\lambda n^\mu)\} \bar{Q}(0, x^-, \mathbf{0_T}) | 0 > \\
D_{H/G}(z) &= \frac{-z}{32\pi k^+} \int dx^- e^{-iP^+ x^- / z} < 0 | G^{b,+\nu}(0) \\
&\quad \{\bar{P} \exp\{-ig_s \int_0^\infty d\lambda n \cdot G(\lambda n^\mu)\}\}^{bc} a_H^\dagger(P^+, \mathbf{0_T}) a_H(P^+, \mathbf{0_T}) \\
&\quad \{P \exp\{ig_s \int_{x^-}^\infty d\lambda n \cdot G(\lambda n^\mu)\}\}^{cd} G_{\nu}^{d,+}(0, x^-, \mathbf{0_T}) | 0 >
\end{aligned} \tag{1}$$

Where $G_\mu(x) = G_\mu^a(x) \frac{\lambda^a}{2}$, $G_\mu^a(x)$ is the gluon field and the $\lambda^a (a = 1, \dots, 8)$ are the Gell-

Mann matrices. The subscription T denotes the transpose. $G^{a,\mu\nu}$ is the gluon field strength and $a_H^\dagger(\mathbf{P})$ is the creation operator for the hadron H . The function $D_{H/Q}(z)$ or $D_{H/G}(z)$ are interpreted as the probability of a quark Q or a gluon G with momentum k to decay into the hadron H with momentum component $P^+ = zk^+$, both are gauge invariant from the definitions. The definitions in (1) are the unrenormalized versions, i.e. all quantities are bare quantities (for renormalization see [5]). When one has a parton, i.e. quark or gluon, instead of the hadron in (1), the perturbative expansion in g_s , the strong coupling constant, can also be expressed with Feymann diagrams, the Feymann rules can be found in [5,6]. The definitions can easily be generalized to hadrons with nonzero spin.

From the definitions a direct calculation is possible if one can express the hadron operators with the parton operators. This is possible for a hadron composed of two heavy quarks, where, as mentioned, the nonrelativistic approach works. To see how to express the hadron operator as a parton one, let us consider a spinless hadron to be a bound-state of a heavy quark Q_1 and an anti-quark \bar{Q}_2 and to have the quantum number 1S_0 . In its rest frame the state can be expressed as:

$$|^1S_0\rangle = A_0 \int \frac{d^3q}{(2\pi)^3} f_0(|\mathbf{q}|) Y_{00}(\theta, \phi) \chi_{s_1 s_2} \frac{1}{\sqrt{3}} \sum_{color} a_{s_1}^\dagger(\mathbf{q}) b_{s_2}^\dagger(-\mathbf{q}) |0\rangle \quad (2)$$

with

$$\chi_{s_1 s_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{s_1 s_2} \quad (3)$$

In Eq(2), $a_{s_1}^\dagger(b_{s_2}^\dagger)$ is the creation operator for $Q_1(\bar{Q}_2)$, the indices s_1 and s_2 are spin-indices, and A_0 is a suitable normalization constant. The function $\psi_0(\mathbf{q}) = f_0(|\mathbf{q}|)Y_{00}(\theta, \phi)$ is obtained through Fourier transformation of the nonrelativistic wavefunction. $f_0(|\mathbf{q}|)$ goes rapidly to zero, when $|\mathbf{q}| \rightarrow \infty$. This means, the region with $|\mathbf{q}| \approx 0$ is dominant in the integral of Eq.(2). In this work we will always take nonzero leading order in $|\mathbf{q}|$ and the hadron mass $M = m_1 + m_2$. Through a Lorenz boost the state can be transformed into a $\mathbf{P} \neq 0$ state:

$$|^1S_0, \mathbf{P}\rangle = A_0 \int \frac{d^3q}{(2\pi)^3} f_0(|\mathbf{q}|) \frac{1}{\sqrt{4\pi}} \chi_{s_1 s_2} \frac{1}{\sqrt{3}} \sum_{color} a_{s_1}^\dagger(\mathbf{p}_1) b_{s_2}^\dagger(\mathbf{p}_2) |0\rangle \quad (4)$$

where

$$p_1 = \frac{m_1}{m_1 + m_2}P + q_p, \quad p_2 = \frac{m_2}{m_1 + m_2}P - q_p \quad (5)$$

q_p is related through the Lorenz boost to q in the rest frame. Now it is easy to show that the bound-state satisfies the normalization condition given in [5] as required for the definitions in (1), if one takes the following form for the hadron creation operator:

$$a_H^\dagger(P) = A_0 \int \frac{d^3q}{(2\pi)^3} f_0(|\mathbf{q}|) \frac{1}{\sqrt{4\pi}} \chi_{s_1 s_2} \frac{1}{\sqrt{3}} \sum_{color} a_{s_1}^\dagger(\mathbf{p}_1) b_{s_2}^\dagger(\mathbf{p}_2) \quad (6)$$

$$A_0^{-1} = \sqrt{\frac{2m_1 m_2}{m_1 + m_2} \cdot \frac{P^0 + P^3}{P^0 + |\mathbf{P}|}}$$

Similarly, one can also obtain the operator for the 3S_1 hadronic state:

$$a_H^\dagger(P, \lambda) = A_0 \int \frac{d^3q}{(2\pi)^3} f_0(|\mathbf{q}|) \frac{1}{\sqrt{4\pi}} (\epsilon_i(\lambda) \cdot \sigma_i \chi)_{s_1 s_2} \frac{1}{\sqrt{3}} \sum_{color} a_{s_1}^\dagger(\mathbf{p}_1) b_{s_2}^\dagger(\mathbf{p}_2) \quad (7)$$

Here $\sigma_i (i = 1, 2, 3)$ are the Pauli matrices and $\epsilon_i(\lambda) (i = 1, 2, 3)$ is the component of the polarization vector of the hadron in its rest frame. λ labels the helicity with respect to the \mathbf{P} direction. With Eq.(6) and Eq.(7) we can now calculate the fragmentation function for hadrons with quantum number 1S_0 and 3S_1 from the definitions in Eq.(1). In the leading order of g_s there are four Feymann diagrams for a quark Q_1 to decay to those hadrons. The diagrams are given in Fig.1, where the double lines represent the Wilson line operators in Eq.(1). One can work in the light cone gauge $n \cdot G = 0$. In this gauge the Wilson line operators in the definitions Eq. (1) disappear and only one of the four diagrams contributes. We will take Feymann gauge. With the Feymann rule in [5,6] one obtains the contribution from each diagram to the fragmentation function $D_{H/Q_1}(z)$. It is straightforward to calculate, and in the calculation one meets the so called spin-sums, which are:

$$u(p_1, s_1) \bar{v}(p_2, s_2) \chi_{s_1 s_2} = \frac{\sqrt{m_1 m_2}}{\sqrt{2}M} \gamma_5 (\gamma \cdot P - M),$$

for $H = ^1S_0$ state

$$u(p_1, s_1) \bar{v}(p_2, s_2) (\epsilon_i(\lambda) \sigma_i \chi)_{s_1 s_2} = -\frac{\sqrt{m_1 m_2}}{\sqrt{2}M} (\gamma \cdot P + M) \epsilon_p(\lambda) \cdot \gamma,$$

for $H = ^3S_1$ state

(8)

Here $\epsilon_p(\lambda)$ is the 4-vector for the polarization of H in the $\mathbf{P} \neq 0$ frame. Similar results for the spin sums in Eq.(8) and below can also be found in [7]. After a straightforward calculation we obtain for the $H = {}^1S_0$ bound-state:

$$D_{H/Q_1}(z) = \frac{2}{81\pi} \frac{|R_0(0)|^2}{m_2^3} \alpha_s^2 \frac{y_2 z (1-z)^2}{(1-y_1 z)^6} \cdot \{6 + 18(1-2y_1)z + (68y_1^2 - 62y_1 + 15)z^2 + 2y_1(-18y_1^2 + 17y_1 - 5)z^3 + 3y_1^2(1-2y_1+2y_1^2)z^4\} \quad (9)$$

and for $H = {}^3S_1$ bound-state:

$$D_{H/Q_1}(z, \lambda = 0) = \frac{2}{81\pi} \frac{|R_0(0)|^2}{m_2^3} \alpha_s^2 \frac{y_2 z (1-z)^2}{(1-y_1 z)^6} \cdot \{2 - 2(1+2y_1)z + (15 - 22y_1 + 16y_1^2)z^2 + 2y_1(-5 + 7y_1 - 6y_1^2)z^3 + 3y_1^3(1-2y_1+2y_1^2)z^4\} \quad (10)$$

$$D_{H/Q_1}(z, \lambda = \pm 1) = \frac{2}{81\pi} \frac{|R_0(0)|^2}{m_2^3} \alpha_s^2 \frac{y_2 z (1-z)^2}{(1-y_1 z)^6} \cdot \{2 - 2(1+2y_1)z + (15 - 16y_1 + 10y_1^2)z^2 + 2y_1(y_1 - 5)z^3 + 3y_1^2z^4\}$$

Here $y_i = m_i/M$ for $i = 1, 2$ and $R_0(0)$ is the radial wavefunction at the origin. Comparing the fragmentation functions calculated in previous works[3,4] we find agreement. Exchanging $y_1 \rightarrow y_2$ one can get D_{H/\bar{Q}_2} . If we take $\bar{Q}_2 = \bar{Q}_1$, then we obtain the fragmentation functions for the quarkonia ($Q_1\bar{Q}_1$).

Now we turn to gluon fragmentaion function. If a hadron has $\bar{Q}_2 = \bar{Q}_1$, then a gluon can decay into that hadron at the order of g_s^4 . In this leading order there are also four Feymann diagrams depicted in Fig.2. Since Hadrons are colorless, there are always two gluons attached to the fermion lines to bulid a color singlet. Because charge-conjungation invariance a gluon can not decay at this order into a hadron with the quantum number 3S_1 (i.e. $J^{PC} = 1^{--}$). Computing the contributions from the diagrams we obtain the fragmentaion function for a 1S_0 hadron:

$$D_{H/G}(z) = \frac{\alpha_s^2}{24\pi} \frac{|R_0(0)|^2}{m_1^3} \{(3-2z)z + 2(1-z)\ln(1-z)\} \quad (11)$$

This is also in agreement with that obtained in [2]. With our results calculated directly from Eq.(1) one can start with the definitions to calculate the high order corrections. It should be pointed out that the fragmentation functions calculated here are at the scale $\mu = M$. For the functions at an arbitrary scale μ one needs to solve the corresponding renormalization group equations and use the results in Eq.(9,10,11) as initial conditions.

With Eq.(9,10,11) we complete the calculations for the fragmentation functions of a S -wave meson. The long-distance effect in these functions are included in the nonrelativistic approach through the radial wave functions at the origin, which can be calculated from potential models or can be directly measured in leptonic decays. For a P -wave meson the calculation is lengthy but still straightforward. Considering a meson to be a $J^{PC} = 1^{++}$ bound state of $(Q_1\bar{Q}_1)$, the creation operator for this hadron is:

$$a^\dagger(\mathbf{P}, \lambda) = A_0 \sqrt{\frac{3}{8\pi}} \int \frac{d^3q}{(2\pi)^3} f_1(|\mathbf{q}|) \frac{q_{i_1}}{|\mathbf{q}|} \epsilon_{i_2}(\lambda) (\sigma_{i_3} \chi)_{s_1 s_2} \epsilon_{i_1 i_2 i_3} a_{s_1}^\dagger(p_1) b_{s_2}^\dagger(p_2) \quad (12)$$

and the spin sum used in the calculation:

$$u(p_1, s_1) \bar{v}(p_2, s_2) q_{i_1} \epsilon_{i_2}(\lambda) (\sigma_{i_3} \chi)_{s_1 s_2} \epsilon_{i_1 i_2 i_3} = \frac{1}{\sqrt{2}} \frac{1}{4M^3} (\gamma \cdot p_1 + m_1) \epsilon_{\mu\nu\sigma\rho} P^\mu \epsilon_p^\nu(\lambda) q_p^\sigma \gamma^\rho (\gamma \cdot P - M) (\gamma \cdot p_2 - m_1) \quad (13)$$

At order α_s^2 a gluon can decay into the $J^{PC} = 1^{++}$ meson, the Feymann diagrams are same as in Fig.2. Keeping the leading order at $|\mathbf{q}|$ we obtain:

$$D_{H/G}(z, \lambda = 0) = \frac{\alpha_s^2}{108} \frac{|R'_1(0)|^2}{4\pi m_1^5} \left\{ \frac{z}{(1-z)} (1-z+2z^2) \right\} \\ D_{H/G}(z, \lambda = \pm 1) = \frac{\alpha_s^2}{108} \frac{|R'_1(0)|^2}{4\pi m_1^5} \left\{ \frac{z}{(1-z)} (1-z+z^2) \right\} \quad (14)$$

The $R'_1(0)$ is the derivative of the radial wavefunction at the origin. The results in Eq.(14) are divergent when $z \rightarrow 1$. The reason is that the gluon exchanged between the quarks in Fig.2 can move collinearly along the quarks, if we take the nonzero leading order at $|\mathbf{q}|$. Such singularity is actually well known as the singularity in the zero-binding limit[8], appearing in the calculations of the hadronic decay widths of a P -wave meson. The situation is similar, if we transpose the diagrams in Fig.2 and calculate the gluon distribution

of the hadron. The singularity at $z = 1$ means that the long-distance effect is present not only in the $R'_1(0)$ but also in the remaining parts which we calculated perturbatively and hence it prevents the naive use of perturbation theory for z near to 1. Recent progress[9] on the P -wave hadron decay shows that the long-distance effect can be clearly separated from the short-distance effect by a *new* factorization theorem[9] for the decay, where one realizes that any meson is a superposition of many components:

$$|M \rangle = \Psi_{Q\bar{Q}}|Q\bar{Q} \rangle + \Psi_{Q\bar{Q}G}|Q\bar{Q}G \rangle + \dots \quad (15)$$

and the configuration of $|Q\bar{Q}G \rangle$ plays an important role in the decay. In the hadronic decay of a P -wave meson one should also take account on the $|Q\bar{Q}G \rangle$ configuration and the decay width can be written in the *new* factorization form, where instead of one, two parameters represent the long-distance effect and the remaining parts can be calculated perturbatively without the singularity. To apply this idea in our case, one needs to know the wavefunction $\Psi_{Q\bar{Q}G}$ in Eq.(15), which can not be obtained through the nonrelativistic approach. A detailed study is needed. Here we recall a phenomenological treatment for the singularity at $z = 1$. Similar treatment of the decay was also used in [8,10]. In calculating the contribution from the diagrams to $D_{H/G}$ we integrated the transverse momentum $|\mathbf{p}_T|$, which is carried by the exchanged gluon, from zero to infinity. However, the meson is not a point-like particle and it has a spatial extension. The extension in the transverse direction is of the order $R \approx 1/M$. If $|\mathbf{p}_T|$ is smaller than $1/R$, one should treat the gluon as a part of the meson. From this point of view, the integral over $|\mathbf{p}_T|$ should be from a nonzero $|\mathbf{p}_T|_{\min} \approx 1/R$ to ∞ . We introduce a parameter β :

$$|\mathbf{p}_T|_{\min} = \beta M \quad (16)$$

If the picture given above is correct, β should be around 1. Here we take it as a free

parameter. With Eq. (16) we obtain:

$$\begin{aligned}
D_{H/G}(z, \lambda = 0) &= \frac{\alpha_s^2}{108} \frac{|R'_1(0)|^2}{4\pi m_1^5} ((1-z)^2 + z^2 \beta^2)^{-3} \\
&\quad \cdot \{z(-2z^7 + 11z^6 - 26z^5 + 35z^4 - 30z^3 + 17z^2 - 6z + 1) \\
&\quad + 3\beta^2 z^4(-2z^4 + 7z^3 - 9z^2 + 5z - 1) + 3\beta^4 z^5(-2z^3 + 4z^2 - 3z + 1)\} \\
D_{H/G}(z, \lambda = \pm 1) &= \frac{\alpha_s^2}{108} \frac{|R'_1(0)|^2}{4\pi m_1^5} ((1-z)^2 + z^2 \beta^2)^{-3} \\
&\quad \cdot \{z(-z^7 + 6z^6 - 16z^5 + 25z^4 - 25z^3 + 16z^2 - 6z + 1) \\
&\quad + 3\beta^2 z^3(-z^5 + 4z^4 - 7z^3 + 7z^2 - 4z + 1)\}
\end{aligned} \tag{17}$$

The results in Eq. (17) are finite at $z = 1$ and for $z \rightarrow 0$ the expressions approach those given in Eq. (14). In Eq.(17) there are two parameters which need to be determined, the $R'_1(0)$ can be obtained from potential models or measured elsewhere, while β is unknown in principle. However, in the fragmentation functions the dependence on the momentum fraction z is predicted by the perturbative calculations. It should be stressed that it is important to know the fragmentation functions for the P -wave meson or quarkonia. If we take Q_1 as a c quark, the quarkonia then corresponds to a χ_{c1} meson and χ_{c1} can substantially decay into J/ψ via radiative decay. Hence, a theoretical prediction of χ_{c1} production via fragmentation is very important for the prediction for the J/ψ production(for example see [11] and the references cited there).

The definitions given in Eq.(1) are for unpolarized partons. One can generalize the definitions to polarized partons and then obtain fragmentation functions for a polarized parton. In this way one can study through heavy meson production the polarization properties of the quarks and gluons produced in high energy collisions, where one expects for large P_t processes that the heavy mesons are produced dominantly via fragmentation[2].

To summarize: We have shown in this letter that the fragmentation functions for a S -wave meson composed of two heavy quarks can be calculated directly from the definitions given by Collins and Soper in [5] and the results obtained are same as those obtained by

other means. A new fragmentation function for a P -wave meson is also obtained, but in contrast to a S -wave meson, there is an extra parameter which needs to be estimated. From the naive picture given above this parameter should be around 1. Finally within the definitions one can conveniently study higher order corrections to the fragmentation functions.

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Figure Captions

Fig.1. The Feymann diagrams for a quark to decay into a hadron.

Fig.2. The Feymann diagrams for a gluon to decay into a hadron.

Fig. 1

Fig. 2

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